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THE EXISTENCE OF d-POLYTOPES WHOSE FACETS HAVE PRESCRIBED AREAS

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ABSTRACT

Three positive numbers $a(1), a(2), a(3)$ are the lengths of the sides of some triangle if, and only if, $a(i_1)+a(i_2)< a(i_3)$ for each permutation i_1, i_2, i_3 of $1, 2, 3$. Here we extend this result to the areas of facets of d-polytopes.

Introduction

Let P be a d-polytope with $n+1$ facets of areas $A_1,\cdots,A_{n+1}$ respectively. Then, by projecting P onto the hyperplane spanned by the ith face,

$$2A_i < \sum_{j=1}^{n+1} A_j, \quad \text{for} \quad i = 1,\cdots,n+1.$$

G. Purdy [2] has asked whether or not the converse holds, i.e., if $A_1,\cdots,A_{n+1}$ are positive numbers with

$$2A_i < \sum_{j=1}^{n+1} A_j, \quad \text{for} \quad i = 1,\cdots,n+1,$$

does there exist in each E^d, $2 \leqq d \leqq n$, a d-polytope P with $n+1$ facets of areas $A_1,\cdots,A_{n+1}$ respectively? The purpose of this note is to prove that this is true.

THEOREM. *Suppose $A_1,\cdots,A_{n+1}$ are positive numbers with $2A_i < \sum_{j=1}^{n+1} A_j$, $i = 1,\cdots,n+1$. Then in any E^d, $2 \leqq d \leqq n$, there exists a d-polytope P with $n+1$ facets $F_1,\cdots,F_{n+1}$ such that the $d-1$ dimensional area of F_i is A_i, $i = 1,\cdots,n+1$.*

LEMMA (Minkowski [1]). *Let $\mathbf{a}_1,\cdots,\mathbf{a}_{n+1}$ be non-zero vectors, whose directions are distinct, which span E^d and suppose $\mathbf{a}_1+\cdots+\mathbf{a}_{n+1}=0$. Then there exists a d-polytope P in E^d with $n+1$ facets $F_1,\cdots,F_{n+1}$ such that $\mathbf{a}_i/|\mathbf{a}_i|$ is the outward*

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normal to F_i and $|a_i|$ is the $d-1$ dimensional area of F_i, $i=1,\cdots,n+1$. Conversely, if P is a d-polytope in E^d with facets $F_1,\cdots,F_{n+1}$ and outward normals $\boldsymbol{u}_1,\cdots,\boldsymbol{u}_{n+1}$ then $|F_1|\boldsymbol{u}_1,\cdots,|F_{n+1}|\boldsymbol{u}_{n+1}$ span E^d and

$$|F_1|\boldsymbol{u}_1+\cdots+|F_{n+1}|\boldsymbol{u}_{n+1}=\boldsymbol{0}.$$

PROOF OF THE THEOREM. If $d=2$ then k can be chosen, $1<k\leqq n$ so that

$$A_1+\cdots+A_{k-1}\leqq A_{k+1}+\cdots+A_{n+1},$$

$$A_1+\cdots+A_k>A_{k+1}+\cdots+A_{n+1}.$$

Consider a triangle with sides of length A_k, $A_1+\cdots+A_{k-1}$, $A_{k+1}+\cdots+A_{n+1}$ respectively. By breaking the sides (outwards) at the required points, the required example is obtained.

So it may be supposed that $d\geqq 3$ and we may suppose that $A_1\leqq A_2\leqq\cdots\leqq A_{n+1}$. We assert that there exist unit vectors $\boldsymbol{u}_1,\cdots,\boldsymbol{u}_n$; $\boldsymbol{v}_1,\cdots,\boldsymbol{v}_n$ in general position such that

$$\left|\sum_{i=1}^{n}A_i\boldsymbol{u}_i\right|>A_{n+1},\tag{1}$$

$$\left|\sum_{i=1}^{n}A_i\boldsymbol{v}_i\right|<A_{n+1}.\tag{2}$$

Since $\Sigma_{i=1}^{n}A_i>A_{n+1}$, (1) is established by slightly perturbing each term of the vector $A_1\boldsymbol{u}+\cdots+A_n\boldsymbol{u}$ for some unit vector $\boldsymbol{u}$ and (2) is established by slightly perturbing each term of $A_n\boldsymbol{u}-A_{n-1}\boldsymbol{u}+\cdots+(-1)^{n+1}A_1\boldsymbol{u}$ for some unit vector $\boldsymbol{u}$.

On the sphere S^{d-1} we choose paths P_i between $\boldsymbol{u}_i$ and $\boldsymbol{v}_i$ which do not pass through $\boldsymbol{u}_j$, $\boldsymbol{v}_j$, $j\neq i$. If $\boldsymbol{z}_i$ is a point on P_i consider the vector

$$w(\boldsymbol{z}_i)=A_i\boldsymbol{z}_i+\sum_{j=1}^{i-1}A_j\boldsymbol{u}_j+\sum_{j=i+1}^{n}A_j\boldsymbol{v}_j.$$

In view of (1), (2), $|w(\boldsymbol{u}_1)|<A_{n+1}$, $|w(\boldsymbol{u}_n)|>A_{n+1}$. Consequently there exists i and $\boldsymbol{z}_i^*$ such that

$$|w(\boldsymbol{z}_i^*)|=A_{n+1}.$$

Some difficulty arises in ensuring that the vectors $-w(\boldsymbol{z}_i^*)$, $A_i\boldsymbol{z}_i^*$, $A_i\boldsymbol{u}_1,\cdots,A_{i-1}\boldsymbol{u}_{i-1}$, $A_{i+1}\boldsymbol{v}_{i+1},\cdots,A_n\boldsymbol{v}_n$ are distinct and span E^d. This may be ensured by arranging that $w(\boldsymbol{z}_i)$ is never antipodal to any of $\boldsymbol{u}_1,\cdots,\boldsymbol{u}_n$; $\boldsymbol{v}_1,\cdots,\boldsymbol{v}_n$; $\boldsymbol{z}_i$ or, if $n=d$, that $|w(\boldsymbol{z}_i)|\neq A_{d+1}$ when $\boldsymbol{z}_i$ is in the $d-1$ subspace

$\mathrm{lin}\{\boldsymbol{u}_1, \cdots, \boldsymbol{u}_{i-1}, \boldsymbol{v}_{i+1}, \cdots, \boldsymbol{v}_d\}$. To arrange this we slightly deviate from the path P_i if necessary.

Then, using the vectors $-w(z_i^*)$, $A_i z_i^*$, $A_1\boldsymbol{u}_1, \cdots, A_{i-1}\boldsymbol{u}_{i-1}$, $A_{i+1}\boldsymbol{v}_{i+1}, \cdots, A_n\boldsymbol{v}_n$ in the lemma completes the proof of the theorem.

References

1. H. Minkowski, *Allgemeine Lehrsätze über die konvexe Polyheder*, in Nachr. Ges. Wiss. Göttingen, 1897, pp. 198–219.

2. G. Purdy, *Spheres tangent to all the faces of a simplex*, J. Combinatorial Theory **17** (1974), 131–133.

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